

This is the calculation of the Klein-Gordon Hamiltonian when we postulate fields that may be written as a normal mode expansion of independent oscillations.

We begin with the Hamiltonian having the determined form

$$H = \int d^3x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\vec{\nabla}\phi)^2 + \frac{1}{2} m^2 \phi^2 \right]$$

The field term we need to use are

$$\pi^2 = - \iint \frac{d^3p d^3p'}{(2\pi)^6} \frac{\sqrt{ww'}}{2} \begin{bmatrix} a(\vec{p}) a(\vec{p}') e^{i(\vec{p}+\vec{p}') \cdot \vec{x}} & -a(\vec{p}) a^\dagger(\vec{p}') e^{i(\vec{p}-\vec{p}') \cdot \vec{x}} \\ -a^\dagger(\vec{p}) a(\vec{p}') e^{-i(\vec{p}-\vec{p}') \cdot \vec{x}} & +a^\dagger(\vec{p}) a^\dagger(\vec{p}') e^{-i(\vec{p}+\vec{p}') \cdot \vec{x}} \end{bmatrix}$$

$$(\vec{\nabla}\phi)^2 = \iint \frac{d^3p d^3p'}{(2\pi)^6 2\sqrt{ww'}} \begin{bmatrix} i(\vec{p}+\vec{p}') \cdot \vec{x} & i(\vec{p}-\vec{p}') \cdot \vec{x} \\ -i(\vec{p}-\vec{p}') \cdot \vec{x} & -i(\vec{p}+\vec{p}') \cdot \vec{x} \end{bmatrix} \begin{bmatrix} -a(\vec{p}) a(\vec{p}') (\vec{p} \cdot \vec{p}') e^{i(\vec{p}+\vec{p}') \cdot \vec{x}} & +a(\vec{p}) a^\dagger(\vec{p}') (\vec{p} \cdot \vec{p}') e^{i(\vec{p}-\vec{p}') \cdot \vec{x}} \\ +a^\dagger(\vec{p}) a(\vec{p}') (\vec{p} \cdot \vec{p}') e^{-i(\vec{p}-\vec{p}') \cdot \vec{x}} & -a^\dagger(\vec{p}) a^\dagger(\vec{p}') (\vec{p} \cdot \vec{p}') e^{-i(\vec{p}+\vec{p}') \cdot \vec{x}} \end{bmatrix}$$

$$\phi^2 = \iint \frac{d^3p d^3p'}{(2\pi)^6 2\sqrt{ww'}} \begin{bmatrix} i(\vec{p}+\vec{p}') \cdot \vec{x} & i(\vec{p}-\vec{p}') \cdot \vec{x} \\ -i(\vec{p}-\vec{p}') \cdot \vec{x} & -i(\vec{p}+\vec{p}') \cdot \vec{x} \end{bmatrix} \begin{bmatrix} a(\vec{p}) a(\vec{p}') e^{i(\vec{p}+\vec{p}') \cdot \vec{x}} & +a(\vec{p}) a^\dagger(\vec{p}') e^{i(\vec{p}-\vec{p}') \cdot \vec{x}} \\ +a^\dagger(\vec{p}) a(\vec{p}') e^{-i(\vec{p}-\vec{p}') \cdot \vec{x}} & +a^\dagger(\vec{p}) a^\dagger(\vec{p}') e^{-i(\vec{p}+\vec{p}') \cdot \vec{x}} \end{bmatrix}$$

$$H = \int d^3x \left\{ \left[\int \frac{d^3p d^3p'}{(2\pi)^6} \frac{\sqrt{ww'}}{4} \left[-a(\vec{p})a(\vec{p}')e^{i(\vec{p}+\vec{p}') \cdot \vec{x}} + a(\vec{p})a^\dagger(\vec{p}')e^{i(\vec{p}-\vec{p}') \cdot \vec{x}} \right. \right. \right. \\ \left. \left. \left. + a^\dagger(\vec{p})a(\vec{p}')e^{-i(\vec{p}-\vec{p}') \cdot \vec{x}} - a^\dagger(\vec{p})a^\dagger(\vec{p}')e^{-i(\vec{p}+\vec{p}') \cdot \vec{x}} \right] \right. \right. \\ \left. \left. + \int \frac{d^3p d^3p'}{(2\pi)^6} \frac{(\vec{p} \cdot \vec{p}')}{4\sqrt{ww'}} \left[-a(\vec{p})a(\vec{p}')e^{i(\vec{p}+\vec{p}') \cdot \vec{x}} + a(\vec{p})a^\dagger(\vec{p}')e^{i(\vec{p}-\vec{p}') \cdot \vec{x}} \right. \right. \right. \\ \left. \left. \left. + a^\dagger(\vec{p})a(\vec{p}')e^{-i(\vec{p}-\vec{p}') \cdot \vec{x}} - a^\dagger(\vec{p})a^\dagger(\vec{p}')e^{-i(\vec{p}+\vec{p}') \cdot \vec{x}} \right] \right. \right. \\ \left. \left. + \left[\int \frac{d^3p d^3p'}{(2\pi)^6} \frac{m^2}{4\sqrt{ww'}} \left[a(\vec{p})a(\vec{p}')e^{i(\vec{p}+\vec{p}') \cdot \vec{x}} + a(\vec{p})a^\dagger(\vec{p}')e^{i(\vec{p}-\vec{p}') \cdot \vec{x}} \right. \right. \right. \right. \\ \left. \left. \left. \left. + a^\dagger(\vec{p})a(\vec{p}')e^{-i(\vec{p}-\vec{p}') \cdot \vec{x}} + a^\dagger(\vec{p})a^\dagger(\vec{p}')e^{-i(\vec{p}+\vec{p}') \cdot \vec{x}} \right] \right] \right\}$$

$$H = \int d^3x \left\{ \left[\int \frac{d^3p d^3p'}{(2\pi)^6} \frac{\sqrt{ww'}}{4} \left[-a(\vec{p})a(\vec{p}')e^{i(\vec{p}+\vec{p}') \cdot \vec{x}} + a(\vec{p})a^\dagger(\vec{p}')e^{i(\vec{p}-\vec{p}') \cdot \vec{x}} \right. \right. \right. \\ \left. \left. \left. + a^\dagger(\vec{p})a(\vec{p}')e^{-i(\vec{p}-\vec{p}') \cdot \vec{x}} - a^\dagger(\vec{p})a^\dagger(\vec{p}')e^{-i(\vec{p}+\vec{p}') \cdot \vec{x}} \right] \right. \right. \\ \left. \left. + \int \frac{d^3p d^3p'}{(2\pi)^6} \frac{(-\vec{p} \cdot \vec{p}' + m^2)}{4\sqrt{ww'}} \left[a(\vec{p})a(\vec{p}')e^{i(\vec{p}+\vec{p}') \cdot \vec{x}} + (\vec{p} \cdot \vec{p}' + m^2)a(\vec{p})a^\dagger(\vec{p}')e^{i(\vec{p}-\vec{p}') \cdot \vec{x}} \right. \right. \right. \\ \left. \left. \left. + (-\vec{p} \cdot \vec{p}' + m^2)a^\dagger(\vec{p})a(\vec{p}')e^{-i(\vec{p}-\vec{p}') \cdot \vec{x}} + (-\vec{p} \cdot \vec{p}' + m^2)a^\dagger(\vec{p})a^\dagger(\vec{p}')e^{-i(\vec{p}+\vec{p}') \cdot \vec{x}} \right] \right] \right\}$$

Integrate over ω .

$$H = \iint \frac{d\vec{p} d\vec{p}'}{(2\pi)^6} \frac{\sqrt{\omega\omega'}}{4} \left[-a(\vec{p})a(\vec{p}') (2\pi)^3 \delta^3(\vec{p}-\vec{p}') + a(\vec{p})a^\dagger(\vec{p}') (2\pi)^3 \delta^3(\vec{p}-\vec{p}') \right. \\ \left. + a^\dagger(\vec{p})a(\vec{p}') (2\pi)^3 \delta^3(\vec{p}-\vec{p}') - a^\dagger(\vec{p})a^\dagger(\vec{p}') (2\pi)^3 \delta^3(\vec{p}+\vec{p}') \right] \\ + \iint \frac{d\vec{p} d\vec{p}'}{(2\pi)^6} \frac{1}{4\sqrt{\omega\omega'}} \left[(-\vec{p}\cdot\vec{p}' + m^2) a(\vec{p})a(\vec{p}') (2\pi)^3 \delta^3(\vec{p}+\vec{p}') + (\vec{p}\cdot\vec{p}' + m^2) a(\vec{p})a^\dagger(\vec{p}') (2\pi)^3 \delta^3(\vec{p}-\vec{p}') \right. \\ \left. + (\vec{p}\cdot\vec{p}' + m^2) a^\dagger(\vec{p})a(\vec{p}') (2\pi)^3 \delta^3(\vec{p}-\vec{p}') + (-\vec{p}\cdot\vec{p}' + m^2) a^\dagger(\vec{p})a^\dagger(\vec{p}') (2\pi)^3 \delta^3(\vec{p}+\vec{p}') \right]$$

$$H = \int \frac{d\vec{p}}{(2\pi)^3} \frac{\omega}{4} \left[-a(\vec{p})a(-\vec{p}) + a(\vec{p})a^\dagger(-\vec{p}) + a^\dagger(\vec{p})a(\vec{p}) - a^\dagger(\vec{p})a^\dagger(-\vec{p}) \right] \\ + \int \frac{d\vec{p}}{(2\pi)^3} \frac{1}{4\omega} \left[(p^2 + m^2) a(\vec{p})a(\vec{p}) + (p^2 + m^2) a(\vec{p})a^\dagger(\vec{p}) \right. \\ \left. + (p^2 + m^2) a^\dagger(\vec{p})a(\vec{p}) + (p^2 + m^2) a^\dagger(\vec{p})a^\dagger(-\vec{p}) \right]$$

$$H = \int \frac{d\vec{p}}{(2\pi)^3} \frac{\omega}{4} \left[-a(\vec{p})a(-\vec{p}) + a(\vec{p})a^\dagger(-\vec{p}) + a^\dagger(\vec{p})a(\vec{p}) - a^\dagger(\vec{p})a^\dagger(-\vec{p}) \right. \\ \left. + a(\vec{p})a(-\vec{p}) + a(\vec{p})a^\dagger(-\vec{p}) + a^\dagger(\vec{p})a(\vec{p}) + a^\dagger(\vec{p})a^\dagger(-\vec{p}) \right]$$

$$H = \int \frac{d\vec{p}}{(2\pi)^3} \frac{\omega}{2} \left[a(\vec{p})a^\dagger(-\vec{p}) + a^\dagger(\vec{p})a(-\vec{p}) \right]$$

$$H = \int \frac{d^3 p}{(2\pi)^3} \frac{\omega}{2} \left[a(\vec{p}) a^\dagger(\vec{p}) + a^\dagger(\vec{p}) a(\vec{p}) - a(\vec{p}) a^\dagger(\vec{p}) + a(\vec{p}) a^\dagger(\vec{p}) \right.$$

$$\left. - a^\dagger(\vec{p}) a(\vec{p}) + a^\dagger(\vec{p}) a(\vec{p}) \right]$$

$$H = \int \frac{d^3 p}{(2\pi)^3} \frac{\omega}{2} \left[2 a^\dagger(\vec{p}) a(\vec{p}) + [a(\vec{p}), a^\dagger(\vec{p})] \right]$$

$$H = \int \frac{d^3 p}{(2\pi)^3} \omega \left[a^\dagger(\vec{p}) a(\vec{p}) + \frac{1}{2} [a(\vec{p}), a^\dagger(\vec{p})] \right]$$